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Volume comparison and the σ_k-Yamabe problem

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Abstract

In this paper we study the problem of finding a conformal metric with the property that the kth elementary symmetric polynomial of the eigenvalues of its Weyl–Schouten tensor is constant. A new conformal invariant involving maximal volumes is defined, and this invariant is then used in several cases to prove existence of a solution, and compactness of the space of solutions (provided the conformal class admits an *admissible* metric). In particular, the problem is completely solved in dimension four, and in dimension three if the manifold is not simply connected.

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1. Introduction

Let (M^n, g) be a smooth, closed Riemannian manifold of dimension n. We denote the Riemannian curvature tensor by *Riem*, the Ricci tensor by *Ric*, and the scalar curvature by R. In addition, the *Weyl–Schouten tensor* is defined by

$$A = \frac{1}{(n-2)}\left(Ric - \frac{1}{2(n-1)} Rg\right). \tag{1.1}$$

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Note that under the action of $O(n)$ the curvature tensor can be decomposed as

$$Riem = W + A \odot g, \tag{1.2}$$

where W denotes the Weyl curvature tensor, and $\odot$ the Kulkarni–Nomizu product [2]. Since the Weyl tensor is conformally invariant, an important consequence of decomposition (1.2) is that the transformation of the Riemannian curvature tensor under conformal deformations of metric is completely determined by the transformation of the symmetric (0,2)-tensor A.

In [24], the second author initiated the study of the fully nonlinear equations arising from the transformation of A under conformal deformations. More precisely, let $g_u = e^{-2u}g$ denote a conformal metric, and consider the equation

$$\sigma_k^{1/k}(g_u^{-1}A_u) = f(x), \tag{1.3}$$

where $\sigma_k : \mathbf{R}^n \to \mathbf{R}$ denotes the elementary symmetric polynomial of degree k, A_u denotes the Weyl–Schouten tensor with respect to the metric g_u, and $\sigma_k^{1/k}(g_u^{-1}A_u)$ means $\sigma_k(\cdot)$ applied to the eigenvalues of the $(1,1)$-tensor $g_u^{-1}A_u$ obtained by "raising an index" of A_u.

To simplify our formulas we usually interpret A_u as a bilinear form on the tangent space with inner product g (instead of g_u). That is, once we fix a background metric g, $\sigma_k(A_u)$ means $\sigma_k(\cdot)$ applied to the eigenvalues of the $(1,1)$-tensor $g^{-1}A_u$. To understand the practical effect of this convention, recall that A_u is related to A by the formula

$$A_u = A + \nabla^2 u + du \otimes du - \tfrac{1}{2}|\nabla u|^2 g \tag{1.4}$$

(see [24]). Consequently, (1.3) is equivalent to

$$\sigma_k^{1/k}\left(A + \nabla^2 u + du \otimes du - \tfrac{1}{2}|\nabla u|^2 g\right) = f(x)e^{-2u}. \tag{1.5}$$

Note that when $k = 1$, then $\sigma_1(g^{-1}A) = trace(A) = \frac{1}{2(n-1)}R$. Therefore, (1.5) is the classical problem of prescribing scalar curvature. This equation is semilinear elliptic; however, when $k > 1$ Eq. (1.5) is fully nonlinear but not necessarily elliptic.

To explain the ellipticity properties of (1.5), following Gårding [9] and Cafarelli–Nirenberg–Spruck [4] we let $\Gamma_k^+ \subset \mathbf{R}^n$ denote the component of $\{x \in \mathbf{R}^n \mid \sigma_k(x) > 0\}$ containing the positive cone $\{x \in \mathbf{R}^n \mid x_1 > 0, \ldots, x_n > 0\}$. In terms of the cones Γ_k^+, ellipticity can be characterized in the following manner (see [24]): If the eigenvalues of $A = A_g$ are everywhere in Γ_k^+, and if u is a solution to (1.5), then u is an elliptic solution. This fact is a consequence of the convexity of the cones Γ_k^+. Following the usual practice, we will say that a metric g is *k-admissible* if the eigenvalues of $A = A_g$ are in Γ_k^+, and we then write $g \in \Gamma_k^+(M^n)$.

The general problem of solving (1.5) with $f(x) = constant$ is referred to as the σ_k-*Yamabe problem*. It will be convenient to normalize the value of this constant, so that the round metric on the sphere is a solution (with no need of rescaling):

$$\sigma_k^{1/k}\left(A + \nabla^2 u + du \otimes du - \tfrac{1}{2}|\nabla u|^2 g\right) = \sigma_k^{1/k}(S^n)e^{-2u}, \tag{1.6}$$

where $\sigma_k^{1/k}(S^n) = \sigma_k^{1/k}(A_0)$, and A_0 is the Weyl–Schouten tensor of the round metric on S^n. We remark that the associated equation is variational when $k = 1$ or 2, but in general not when $k > 2$ (see [24]).

The variational approach to the classical Yamabe problem lead to the definition of the *Yamabe invariant* $Y(M^n, [g])$ of a conformal class of metrics:

$$Y(M^n, [g]) \equiv \inf_{\tilde{g} \in [g]} (vol(\tilde{g}))^{-(n-2)/n} \int R_{\tilde{g}}\, dvol_{\tilde{g}}. \tag{1.7}$$

It is a result of Aubin that $Y(M^n, [g]) \leqslant Y(S^n, g_0)$, where g_0 denotes the round metric, and when strict inequality holds, existence and compactness of solutions can be easily established (see [18]). Thus, the resolution of the classical Yamabe problem is equivalent to the result, due in some cases to Aubin [1] and in the remaining cases to Schoen [23], that equality holds only when the manifold is conformally equivalent to the sphere.

Our first goal in this paper is to define a new conformal invariant associated to Eq. (1.5) when $k \geqslant n/2$.

Definition 1. Let (M^n, g) be a compact n-dimensional Riemannian manifold. For $n/2 \leqslant k \leqslant n$ we define the *k-maximal volume* of $[g]$ by

$$\Lambda_k(M^n, [g]) = \sup\{vol(e^{-2u}g) \mid e^{-2u}g \in \Gamma_k^+(M^n) \text{ with } \sigma_k^{1/k}(g_u^{-1}A_u) \geqslant \sigma_k^{1/k}(S^n)\}. \tag{1.8}$$

If $[g]$ does not admit a k-admissible metric, we set $\Lambda_k(M^n, [g]) = +\infty$.

By recent work of the second author with Guan and Wang [10], a k-admissible metric g with $k > n/2$ necessarily has positive Ricci curvature. In fact, their result is quantitative, in the sense that once we make the normalization $\sigma_k^{1/k}(g^{-1}A_g) \geqslant \sigma_k^{1/k}(S^n)$ a (sharp) lower bound for the Ricci curvature is given (see Section 4). Using Bishop's inequality, it follows that the invariant Λ_k is nontrivial when $k > n/2$:

Proposition 1.1. *If* $[g]$ *admits a k-admissible metric with* $k > n/2$, *then there is a constant* $C = C(n)$ *such that* $\Lambda_k(M^n, [g]) < C(n)$.

When $k = n/2$ the situation is more complicated. For example, if (M^n, g) is locally conformally flat (LCF) and n is even, then the integral

$$\int_{M^n} \sigma_{n/2}(g^{-1}A)\, dvol \tag{1.9}$$

is conformally invariant; see [24]. Therefore, if $g \in \Gamma^+_{n/2}(M^n)$ satisfies $\sigma^{2/n}_{n/2}(g^{-1}A) \geqslant \sigma^{2/n}_{n/2}(S^n)$, then

$$\int_{M^n} \sigma_{n/2}(g^{-1}A)\, dvol \geqslant \sigma_{n/2}(S^n) vol(g).$$

Consequently, the maximal volume of $[g]$ is finite. In fact, we can say more: since the assumption of k-admissibility with $k > n/2$ already implies that the Ricci curvature is positive, if (M^n, g) is LCF then by Kuiper's theorem [17] it must be a space form. Since the dimension is even, by Synge's theorem (M^n, g) is conformally equivalent to S^n or $\mathbf{RP}^n$. Finally, by Proposition 8 in [24] and the Chern–Gauss–Bonnet formula it follows that $\Lambda_{n/2}(M^n, [g]) = vol(S^n)$ or $\Lambda_{n/2}(M^n, [g]) = \frac{1}{2} vol(S^n)$, depending on whether (M^n, g) is conformally equivalent to the sphere or projective space.

In four dimensions integral (1.9) is *always* conformally invariant, so the preceding argument can be applied to show the finiteness of $\Lambda_2(M^4, [g])$ for any conformal four-manifold which admits a 2-admissible metric (see Theorem 1.4 below for a sharp version of this result). In general, however, it is unclear whether $\Lambda_{n/2}$ is finite.

In analogy with the classical Yamabe problem, when our invariant is strictly less than the value obtained by the round metric on the sphere we obtain existence of solutions to (1.6):

Theorem 1.1. *Let (M^n, g) be a closed n-dimensional Riemannian manifold satisfying*

$$\Lambda_k(M^n, [g]) < vol(S^n), \tag{1.10}$$

where $vol(S^n)$ denotes the volume of the round sphere. Then $[g]$ admits a solution $g_u = e^{-2u} g$ of (1.6). *Furthermore, the set of solutions of* (1.6) *is compact in the C^m-topology for any $m \geqslant 0$.*

Despite the parallels with the Yamabe problem, Theorem 1.1 can only be considered satisfying if condition (1.10) is known to be sharp. Although we conjecture this to be the case in general, we can only substantiate it in dimensions three and four. In each case the techniques for proving sharp estimates of $\Lambda_k(M^n, [g])$ are quite different in spirit.

In three dimensions our estimate follows from the volume comparison theorem of Bray [3]. We will give a precise statement of his result later; for now we simply state the consequence for our invariant.

Theorem 1.2. *Let (M^3, g) be a closed Riemannian three-manifold, and assume $[g]$ admits a k-admissible metric with $k = 2$ or 3. Then*

$$\Lambda_k(M^3, [g]) \leqslant vol(S^3). \tag{1.11}$$

The proof of this result allows an important refinement of inequality (1.11). As a consequence, we are able to verify the assumptions of Theorem 1.1 whenever M^3 is not simply connected:

Theorem 1.3. *Let (M^3, g) be a closed Riemannian three-manifold, and assume $[g]$ admits a k-admissible metric with $k = 2$ or 3. Let $\pi_1(M^3)$ denote the fundamental group of M^3. Then*

$$\Lambda_k(M^3, [g]) \leqslant \frac{vol(S^3)}{||\pi_1(M^3)||}. \tag{1.12}$$

Corollary 1.1. *Let (M^3, g) be a closed, nonsimply connected Riemannian three-manifold. If g is k-admissible with $k = 2$ or 3, then $[g]$ admits a solution $g_u = e^{-2u}g$ of* (1.6). *Furthermore, the set of solutions of* (1.6) *is compact in the C^m-topology for any $m \geqslant 0$.*

In four dimensions, our estimates of Λ_k follow from the sharp integral estimate for $\sigma_2(A)$ due to the first author [14].

Theorem 1.4. *Let (M^4, g) be a closed Riemannian four-manifold, and assume $[g]$ admits a k-admissible metric with $2 \leqslant k \leqslant 4$. Then*

$$\Lambda_k(M^4, [g]) \leqslant vol(S^4). \tag{1.13}$$

Furthermore, equality holds in (1.13) *if and only if (M^4, g) is conformally equivalent to the round sphere.*

Corollary 1.2. *Let (M^4, g) be a closed Riemannian four-manifold, and assume g is a k-admissible metric with $2 \leqslant k \leqslant 4$. Then $[g]$ admits a solution $g_u = e^{-2u}g$ of* (1.6). *Furthermore, if (M^4, g) is not conformally equivalent to the round sphere, then the set of solutions of* (1.6) *is compact in the C^m-topology for any $m \geqslant 0$.*

When $k = 2$ the result of Corollary 1.2 was established in [5]. Combining Corollary 1.2 with the four-dimensional solution of the Yamabe problem [23], it follows that the σ_k-Yamabe problem is completely solved in four dimensions.

Similar to the three-dimensional case, if we impose certain topological conditions then inequality (1.13) can be sharpened. Since the work of Viaclovsky–Guan–Wang cited above shows that a k-admissible metric with $k > n/2$ has positive Ricci curvature, by the classical Bochner theorem the first de Rham cohomology group $H^1(M^4) = 0$. On the other hand, if the second de Rham cohomology group is nontrivial, then the L^2-estimates of the Weyl curvature tensor in [13] can be used to give sharp estimates of the maximal volume. To state this result, let b^+ (resp., b^-) denote the dimension of the largest subspace of $H^2(M^4)$ on which the intersection form is positive (resp., negative) definite. Let $\chi(M^4)$ denote the Euler characteristic and $\tau(M^4) = b^+ - b^-$ the signature of M^4.

Theorem 1.5. *Let M^4 be a smooth, compact, orientable four-manifold with $b^+ > 0$. If $g \in \Gamma_k^+$ with $2 \leqslant k \leqslant 4$, then*

$$\Lambda_k(M^4, [g]) \leqslant \frac{2}{9}\pi^2(2\chi(M^4) + 3\tau(M^4)). \tag{1.14}$$

In particular, if $2\chi(M^4) + 3\tau(M^4) < 12$, then $\Lambda_k(M^4, [g]) < vol(S^4) = \frac{8}{3}\pi^2$.

Furthermore, equality holds if and only if $[g]$ admits a (positive) Kähler–Einstein metric which attains the maximal volume. In this case, M^4 is diffeomorphic to either $S^2 \times S^2$, $\mathbf{CP}^2$, or $\mathbf{CP}^2 \# m(-\mathbf{CP}^2)$ with $3 \leqslant m \leqslant 8$.

In higher dimensions we do not have a sharp estimate of our invariant. However, the proof of Theorem 1.3 can be adapted to give the following result:

Theorem 1.6. *There is a number N, depending only on k and n, with the following property: if M^n is a closed n-dimensional manifold whose fundamental group satisfies $||\pi_1(M^n)|| > N(k, n)$, then any k-admissible metric g satisfies $\Lambda_k(M^n, [g]) < vol(S^n)$.*

There has been a considerable amount of recent activity devoted to the study of (1.3) with $k > 1$ (see [5–7,11,12,19,20,22,25,26]). With a few notable exceptions, most of these works consider the case where the background metric is k-admissible.

In [26], the second author established global a priori C^1- and C^2-estimates for k-admissible solutions which depend on C^0-estimates. Since (1.5) is a convex function of the eigenvalues of A_u, the work of Evans and Krylov [8,16] give $C^{2,\alpha}$ bounds once C^2 bounds are known. Consequently, one can derive estimates of all orders from classical elliptic regularity, provided C^0-bounds are known.

Subsequently, Guan and Wang [12] proved local versions of these estimates which only depend on a lower bound for solutions. Their estimates will figure prominently in our analysis. Recently, Li and Li [20] proved Harnack estimates for solutions of (1.5), and a classification result for entire solutions on $\mathbf{R}^n$. Their classification result will also be used in the proof of Theorem 1.1.

For global estimates the result of [26] is optimal: since (1.3) is invariant under the action of the conformal group, a priori C^0-bounds may fail for the usual reason (i.e., the conformal group of the round sphere). Some results have managed to distinguish the case of the sphere, thereby giving bounds when the manifold is not conformally equivalent to S^n. For example, [5] proved the existence of solutions to (1.5) when $k = 2$ and g is 2-admissible, for any function $f(x)$, provided (M^4, g) is not conformally equivalent to the sphere. In [26] the second author studied the case $k = n$, and defined another conformal invariant associated to admissible metrics. When this invariant is below a certain value, one can establish C^0-estimates. Using this fact he proved the existence of solutions to (1.6) on a large class of conformal manifolds.

When (M^n, g) is locally conformally flat and k-admissible, the article [19] gives a compactness result for solutions of (1.5) for any $k \geqslant 1$, assuming (M^n, g) is not conformally equivalent to the sphere. Guan and Wang [11] used a parabolic version of (1.6) to prove global existence (in time) of solutions and convergence to a solution of (1.6). As we observed above, the assumption of LCF and k-admissibility with $k \geqslant n/2$ implies that (M^n, g) is conformally equivalent to a space form.

We conclude the introduction with an outline of the paper. In Section 2 we lay the groundwork for solving (1.5) by introducing a one-parameter family of auxiliary equations. This requires us to establish various a priori estimates, which are contained in Sections 2 and 3. These estimates allow us to apply the degree theory for fully nonlinear equations developed by Li [21] to prove the existence of solutions when $\Lambda_k(M^n, [g]) < vol(S^n)$. Finally, in Section 4 we prove some estimates for the conformal invariant $\Lambda_k(M^n, [g])$.

2. The auxiliary equation: local estimates

Let M^n be a closed n-dimensional manifold, and suppose $g \in \Gamma_k^+(M^n)$. By rescaling, we assume that g has unit volume. Consider the equation

$$\sigma_k^{1/k}\left(\lambda_k g + \nabla^2 u + du \otimes du - \tfrac{1}{2}|\nabla u|^2 g\right) = \left(\int e^{-(n+1)u}\right)^{\frac{2}{n+1}}, \tag{2.1}$$

where λ_k is given by

$$\lambda_k = \binom{n}{k}^{-1/k}. \tag{2.2}$$

This choice of λ_k implies $\sigma_k(\lambda_k g) = 1$. Consequently, $u \equiv 0$ is a solution of (2.1).

Lemma 2.1. $u \equiv 0$ *is the unique solution of* (2.1).

Proof. This follows from the maximum principle, as explained in Proposition 5 of [26]. Suppose u is a solution of (2.1). At a point x_0 where u attains its maximum, $\nabla^2 u(x_0)$ is negative semi-definite and $du(x_0) = 0$, so (2.1) implies

$$\begin{aligned}\left(\int e^{-(n+1)u}\right)^{\frac{2}{n+1}} &= \sigma_k^{1/k}(\lambda_k g + \nabla^2 u(x_0)) \\ &\leqslant \sigma_k^{1/k}(\lambda_k g) \\ &= 1.\end{aligned} \tag{2.3}$$

Applying a similar argument at the minimum of u we find

$$\left(\int e^{-(n+1)u}\right)^{\frac{2}{n+1}} \geqslant 1. \tag{2.4}$$

Therefore,

$$\left(\int e^{-(n+1)u}\right)^{\frac{2}{n+1}} = 1. \tag{2.5}$$

By the Newton–Maclaurin inequality,

$$\begin{aligned}
1 &= \left(\int e^{-(n+1)u}\right)^{\frac{2}{n+1}} \\
&= \sigma_k^{1/k}\left(\lambda_k g + \nabla^2 u + du \otimes du - \tfrac{1}{2}|\nabla u|^2 g\right) \\
&\leqslant \frac{1}{n}\binom{n}{k}^{1/k} \sigma_1\left(\lambda_k g + \nabla^2 u + du \otimes du - \tfrac{1}{2}|\nabla u|^2 g\right) \\
&= \frac{1}{n}\binom{n}{k}^{1/k}\left(\lambda_k n + \Delta u - \frac{(n-2)}{2}|\nabla u|^2\right) \\
&= 1 + \frac{1}{n}\binom{n}{k}^{1/k}\left(\Delta u - \frac{(n-2)}{2}|\nabla u|^2\right).
\end{aligned} \tag{2.6}$$

Then the maximum principle implies u is a constant, and (2.5) forces $u \equiv 0$. $\square$

For the next Lemma, define the operator

$$\Psi[u] = \sigma_k^{1/k}\left(\lambda_k g + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g\right) - \left(\int e^{-(n+1)u}\right)^{\frac{2}{n+1}}. \tag{2.7}$$

By Lemma 2.1, $u_0 \equiv 0$ is the unique solution of

$$\Psi[u_0] = 0. \tag{2.8}$$

Let $\mathscr{L}_{u_0}[h] = \frac{d}{ds}\Psi[u_0 + sh]|_{s=0}$ denote the linearization of $\Psi[\cdot]$ at the solution $u = u_0$. Then

$$\mathscr{L}_{u_0}[h] = \gamma_{k,n}\Delta h + 2\int h, \tag{2.9}$$

where $\gamma_{k,n} = (n\lambda_k)^{-1}$.

Lemma 2.2. *$\mathscr{L}_{u_0} : C^{2,\alpha} \to C^{\alpha}$ is invertible.*

Proof. Given $f \in C^{\alpha}$, let h_1 be the unique solution of

$$\gamma_{k,n}\Delta h_1 = f - \bar{f} \tag{2.10}$$

satisfying

$$\bar{h}_1 = 0, \tag{2.11}$$

where bars denote the mean value (recall the background metric has unit volume). If we take $h = h_1 + \frac{1}{2}\bar{f}$, then by (2.10) and (2.11)

$$\begin{aligned}\mathscr{L}_{u_0}[h] &= \gamma_{k,n}\Delta h + 2\int h \\ &= \gamma_{k,n}\Delta h_1 + 2\int\left(h_1 + \frac{1}{2}\bar{f}\right) \\ &= f - \bar{f} + \bar{f} \\ &= f.\end{aligned}$$

Using the maximum principle, it is easy to see that h is in fact the unique solution of $\mathscr{L}_{u_0}[h] = f$. $\square$

We now introduce a one-parameter family of equations connecting Eq. (1.6) with Eq. (2.1). For $t \in [0,1]$, consider

$$\begin{aligned}&\sigma_k^{1/k}\left(\lambda_k(1-\psi(t))g + \psi(t)A + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g\right) \\ &\quad = (1-t)\left(\int e^{-(n+1)u}\right)^{\frac{2}{n+1}} + \psi(t)\sigma_k^{1/k}(S^n)e^{-2u},\end{aligned} \tag{2.12}$$

where $\psi(t) \in C^1[0,1]$ satisfies $0 \leqslant \psi(t) \leqslant 1$, $\psi(0) = 0$, and $\psi(t) \equiv 1$ for $t \geqslant \frac{1}{2}$. From the properties of $\psi(t)$ we see that if u is a solution of (2.12) with $t \geqslant \frac{1}{2}$, then $\sigma_k^{1/k}(A_u) \geqslant \sigma_k^{1/k}(S^n)e^{-2u}$. Therefore,

$$\Lambda_k(M^n,[g]) \geqslant \sup\left\{vol(g_u) \mid u \text{ satisfies (2.12) with } t \geqslant \tfrac{1}{2}\right\}. \tag{2.13}$$

Since (2.12) admits a unique solution when $t = 0$, we would like to use a degree theoretic argument to show that it also admits a solution when $t = 1$. The degree theory developed by Li [21] for second order fully nonlinear equations provides a framework for this approach. We will explain the details in Section 3, but it may help the reader to appreciate the estimates of this section if we first provide an overview of our plan.

The first step is to compute the Leray–Schauder degree of the solution $u \equiv 0$ of (2.1). By Lemmas 2.1 and 2.2 this degree is nonzero. The next step is to appeal to the homotopy invariance of the degree to conclude that (2.12) has a solution when $t = 1$.

To justify this, however, we need to establish a priori bounds for solutions of (2.12). As we shall see, when $t<1$ the integral term in (2.12) imposes L^{∞}-bounds on solutions. By the a priori C^1- and C^2-estimates of [26], along with the aforementioned results of Krylov [16] and Evans [8], such L^{∞}-bounds will imply bounds on derivatives of all orders.

The conformal invariance of Eq. (2.12) when $t=1$ leads to predictable difficulties when deriving estimates with t close to 1. As $t\to 1$, we need to use a standard blow-up procedure in order to show that the assumption $\Lambda_k(M^n,[g])<vol(S^n)$ imposes L^{∞}-bounds on solutions. The classification of solutions of (1.6) on Euclidean space Li and Li [19] will be important in this respect.

With this overview in mind, we begin with a basic estimate for solutions of (2.12) with $t<1$.

Theorem 2.1. *For any fixed* $0<\delta<1$, *there is a constant* $C=C(\delta,g)$ *such that any solution of* (2.12) *with* $t\in[0,1-\delta]$ *satisfies*

$$||u||_{C^{4,\alpha}}\leqslant C. \tag{2.14}$$

Proof. The proof of this estimate is divided into a few intermediate steps, starting with an estimate on the minimum of solutions.

Proposition 2.1. *If* u *is a solution of* (2.12) *with* $t\in[0,1-\delta]$, *then there is a constant* $C=C(\delta,g)$ *such that*

$$\min_{M^n} u\geqslant C. \tag{2.15}$$

Proof. This proposition is essentially a corollary of the ε-regularity result for solutions of (1.3) due to Guan and Wang [12]. However, (1.3) and (2.12) differ by a constant term; thus we need to clarify some estimates to show that their argument still works.

We begin by noting that the integral in (2.12) is uniformly bounded for $t\leqslant 1-\delta$.

Lemma 2.3. *Let* u *be a solution of* (2.12) *with* $t\in[0,1)$. *Then there is a constant* $C=C(g)$ *such that*

$$(1-t)\left(\int e^{-(n+1)u}\right)^{\frac{2}{n+1}}\leqslant C. \tag{2.16}$$

Proof. To see this we apply the maximum principle once again: At a point x_0 where u attains its maximum, $\nabla^2 u(x_0)$ is negative semi-definite and $du(x_0)=0$, so (2.12) implies

$$\begin{aligned}(1-t)\left(\int e^{-(n+1)u}\right)^{\frac{2}{n+1}} &\leqslant (1-t)\left(\int e^{-(n+1)u}\right)^{\frac{2}{n+1}}+\psi(t)\sigma_k^{1/k}(S^n)e^{-2u(x_0)}\\ &=\sigma_k^{1/k}(\lambda_k(1-\psi(t))g+\psi(t)A(x_0)+\nabla^2 u(x_0))\end{aligned}$$

$$\leqslant \sigma_k^{1/k}(\lambda_k(1-\psi(t))g + \psi(t)A(x_0))$$
$$\leqslant C.$$

This proves the Lemma. $\square$

Corollary 2.1. *Let u be a solution of* (2.12) *with $t \leqslant 1-\delta$. Then there is a constant $C = C(\delta, g)$ such that*

$$\int e^{-(n+1)u} \leqslant C. \tag{2.17}$$

We now turn to the proof of (2.15), arguing by contradiction. Suppose to the contrary we have a sequence $\{u_j\}$ of solutions of (2.12) with $t = t_j \leqslant 1-\delta$, and that $\min u_j \to -\infty$. At a point z_j where u_j attains its minimum let $\exp_{z_j}$: $B(0, \iota_0/2) \subset T_{z_j}M^n \approx \mathbf{R}^n \to M^n$ denote the exponential map, where ι_0 is the injectivity radius of (M^n, g). Let ε_j satisfy $\log \varepsilon_j = \min u_j = u_j(z_j)$, and define

$$T_j(x) = \exp_{z_j}(\varepsilon_j x),$$

$$g_j = \varepsilon_j^{-2} T_j^* g,$$

$$\begin{aligned}\tilde{u}_j(x) &= (T_j^* u_j)(x) - \log \varepsilon_j \\ &= u_j(\exp_{z_j}(\varepsilon_j x)) - \log \varepsilon_j.\end{aligned} \tag{2.18}$$

Then each $\tilde{u}_j$ is defined on $B(0, \varepsilon_j^{-1}\iota_0/2) \subset \mathbf{R}^n$ and satisfies $\tilde{u}_j(x) \geqslant 0$, $\tilde{u}_j(0) = 0$. In addition, since u_j satisfies (2.12), $\tilde{u}_j$ satisfies

$$\begin{aligned}&\sigma_k^{1/k}\left((\lambda_k(1-\psi(t_j))g_j + \psi(t_j)A_j) + \nabla^2\tilde{u}_j + d\tilde{u}_j \otimes d\tilde{u}_j - \frac{1}{2}|\nabla\tilde{u}_j|^2 g_j\right) \\ &= \varepsilon_j^2(1-t_j)\left(\int e^{-(n+1)u_j}\right)^{\frac{2}{n+1}} + \psi(t_j)\sigma_k^{1/k}(S^n)e^{-2\tilde{u}_j},\end{aligned} \tag{2.19}$$

where $A_j = A_{g_j}$, and the covariant derivatives in (2.19) are with respect to g_j. Note that g_j converges to the Euclidean metric ds^2 on compact sets in the C^m-topology, for any $m \geqslant 1$.

Next we claim that for any $\rho > 1$, there is a constant $C = C(\rho, g)$ such that

$$\max_{B(0,\rho)} |\nabla \tilde{u}_j|^2 \leqslant C. \tag{2.20}$$

This estimate is a consequence of the local C^1-estimate of Guan and Wang:

Lemma 2.4 (See Guan and Wang [12], Proposition 2). *Let $u \in C^3$ be an admissible solution of*

$$F(u) = \sigma_k^{1/k}\left(A + \nabla^2 u + du \otimes du - \tfrac{1}{2}|\nabla u|^2 g\right) = f(x)e^{-2u} \tag{2.21}$$

on $B(0, 2\rho)$, where $\rho > 0$. Then there is a constant $C(k, n, \rho, ||g||_{C^3(B(0,\rho))}, ||f||_{C^3(B(0,\rho))})$ such that

$$|\nabla u|^2(x) \leqslant C(1 + e^{-2\inf_{B(0,\rho)} u}) \tag{2.22}$$

for all $x \in B(0, \rho/2)$.

In our case, $\tilde{u}_j$ satisfies

$$F(\tilde{u}_j) = \varepsilon_j^2(1 - t_j)\left(\int e^{-(n+1)u_j}\right)^{\frac{2}{n+1}} + \psi(t_j)\sigma_k^{1/k}(S^n)e^{-2\tilde{u}_j}. \tag{2.23}$$

If we imitate the proof of [12], the only necessary changes appear in the estimates of inequality (13) of Proposition 2 in [12]. More specifically, Guan and Wang estimate the term

$$\begin{aligned}\sum_l F_l u_l &= \sum_l e^{-2u}(f_l u_l - 2fu_l^2)\\ &\geqslant -C(1 + e^{-2u})|\nabla u|^2,\end{aligned} \tag{2.24}$$

where the subscript l denotes $\frac{\partial}{\partial x_l}$. Since our definition of F differs only by a constant term, we can literally copy their argument to obtain the same estimate for $\tilde{u}_j$:

$$\max_{B(0,\rho)} |\nabla \tilde{u}_j|^2 \leqslant C(\rho, g, \min \tilde{u}_j). \tag{2.25}$$

Of course, in our case $\tilde{u}_j \geqslant 0$, and so (2.20) follows.

Combining the gradient bound (2.20) with the condition $\tilde{u}_j(0) = 0$ we see that

$$\min_{B(0,1)} e^{\tilde{u}_j} \geqslant C(g) > 0. \tag{2.26}$$

On the other hand, pulling back to M^n by T_j^{-1} and using the integral bound (2.17) we have

$$\begin{aligned}\int_{B(0,1)} e^{-(n+1)\tilde{u}_j}\, dvol_{g_j} &= \varepsilon_j \int_{B(z_j,\varepsilon_j)} e^{-(n+1)u_j}\, dvol_g\\ &\to 0\end{aligned}$$

as $j \to \infty$. Since this contradicts (2.26), we see that the sequence $\{u_j\}$ must be bounded from below. $\square$

Proposition 2.2. *If u is a solution of* (2.12) *with $t \leqslant 1-\delta$, then there is a constant $C = C(\delta, g)$ such that*

$$\max_{M^n} u \leqslant C. \tag{2.27}$$

Proof. As we explained in the proof of Theorem 2.1, the localized gradient estimate of Guan and Wang can be adapted to Eq. (2.12), giving the bound

$$\max_{M^n} |\nabla u| \leqslant C(1 + e^{-2 \min u}) \leqslant C(\delta, g). \tag{2.28}$$

This immediately implies the Harnack inequality

$$\max_{M^n} u \leqslant \min_{M^n} u + C. \tag{2.29}$$

The upper bound (2.27) will be a consequence of the following Lemma:

Lemma 2.5. *If u is a solution of* (2.12) *with $t \in [0,1]$, then there is a constant $C = C(g)$ such that*

$$\min_{M^n} u \leqslant C. \tag{2.30}$$

Proof. Let x_0 be a point at which u attains its minimum. Then

$$(1-t)\left(\int e^{-(n+1)u}\right)^{\frac{2}{n+1}} + \psi(t)\sigma_k^{1/k}(S^n)e^{-2u(x_0)} \leqslant (1 + \sigma_k^{1/k}(S^n))e^{-2 \min u}. \tag{2.31}$$

At x_0, $\nabla^2 u(x_0)$ is positive semi-definite and $du(x_0) = 0$. Therefore,

$$\begin{aligned} (1-t)\left(\int e^{-(n+1)u}\right)^{\frac{2}{n+1}} &+ \psi(t)\sigma_k^{1/k}(S^n)e^{-2u(x_0)} \\ &= \sigma_k^{1/k}(\lambda_k(1-\psi(t))g + \psi(t)A(x_0) + \nabla^2 u(x_0)) \\ &\geqslant \sigma_k^{1/k}(\lambda_k(1-\psi(t))g + \psi(t)A(x_0)). \end{aligned} \tag{2.32}$$

Since $\sigma_k : \Gamma_k^+ \to \mathbf{R}$ is a concave function (see [26], Proposition 1),

$$\begin{aligned} \sigma_k^{1/k}(\lambda_k(1-\psi(t))g + \psi(t)A(x_0)) &\geqslant \sigma_k^{1/k}(\lambda_k(1-\psi(t))g) + \sigma_k^{1/k}(\psi(t)A(x_0)) \\ &= (1-\psi(t)) + \psi(t)\sigma_k^{1/k}(A(x_0)) \\ &\geqslant C(g) > 0. \end{aligned} \tag{2.33}$$

Combining (2.31), (2.32) and (2.33) we find

$$e^{-2\min u} \geqslant C(g) > 0,$$

which implies (2.30). □

To complete the proof of Theorem 2.1 we appeal to the global a priori estimates of [26] (see Propositions 6 and 8): If u is a solution of (2.12) with $0 \leqslant t \leqslant 1-\delta$, then

$$\begin{aligned} \|\nabla u\|_\infty + \|\nabla^2 u\|_\infty &\leqslant C(\|u\|_\infty) \\ &\leqslant C(\delta, g). \end{aligned} \tag{2.34}$$

As explained in the introduction, the work of Evans [8] and Krylov [16] now give bounds on the Holder norms of the second derivatives of u. Hence, the estimate (2.14) follows from classical elliptic regularity. □

3. Global estimates and existence

Having established estimates for solutions of (2.12) when t is bounded away from 1, we now study what happens as $t \to 1$. As the title of this section indicates, the analysis of this case depends on *global* invariants of the manifold—namely, Λ_k—rather than *local* properties of Eq. (2.12).

Theorem 3.1. *Suppose $\Lambda_k(M^n, [g]) < vol(S^n)$. If u is a solution of* (2.12) *with $t \in [0, 1]$, then there is a constant $C = C(g)$ such that*

$$\|u\|_{C^{4,\alpha}} \leqslant C. \tag{3.1}$$

Proof. Like the proof of Theorem 2.1, we use a blow-up argument. However, since the integral bound (2.17) degenerates as $t \to 1$ we can no longer rely on an ε-regularity result. This is to be expected, given the phenomenon of bubbling. In any case, we still begin with an estimate of the lower bound of u.

Proposition 3.1. *There is constant $C = C(g)$ such that*

$$\min u \geqslant -C. \tag{3.2}$$

Proof. Once again, we argue by contradiction: Suppose to the contrary we have a sequence $\{u_j\}$ of solutions of (2.12) with $t = t_j \to 1$, and that $\min u_j \to -\infty$. At a point z_j where u_j attains its minimum let $\exp_{z_j}\colon B(0, \iota_0/2) \subset T_{z_j}M^n \approx \mathbf{R}^n \to M^n$ denote the exponential map, where ι_0 is the injectivity radius of (M^n, g). As before, let ε_j

satisfy $\log \varepsilon_j = \min u_j = u_j(z_j)$, and define

$$T_j(x) = \exp_{z_j}(\varepsilon_j x),$$

$$g_j = \varepsilon_j^{-2} T_j^* g,$$

$$\begin{aligned}\tilde{u}_j(x) &= (T_j^* u_j)(x) - \log \varepsilon_j \\ &= u_j(\exp_{z_j}(\varepsilon_j x)) - \log \varepsilon_j. \end{aligned} \tag{3.3}$$

Then each $\tilde{u}_j$ is defined on $B(0, \varepsilon_j^{-1} \iota_0/2) \subset \mathbf{R}^n$ and satisfies $\tilde{u}_j(x) \geqslant 0,\ \tilde{u}_j(0) = 0$. In addition, by (2.12) $\tilde{u}_j$ satisfies (2.19):

$$\begin{aligned}&\sigma_k^{1/k}\left((\lambda_k(1-\psi(t_j))g_j + \psi(t_j)A_j) + \nabla^2 \tilde{u}_j + d\tilde{u}_j \otimes d\tilde{u}_j - \frac{1}{2}|\nabla \tilde{u}_j|^2 g_j\right) \\ &\quad = \varepsilon_j^2(1-t_j)\left(\int e^{-(n+1)u_j}\right)^{\frac{2}{n+1}} + \psi(t_j)\sigma_k^{1/k}(S^n)e^{-2\tilde{u}_j}. \end{aligned} \tag{3.4}$$

Note that by Lemma 2.3, as $j \to \infty$ the integral term above goes to zero:

$$\begin{aligned}\varepsilon_j^2(1-t_j)\left(\int e^{-(n+1)u_j}\right)^{\frac{2}{n+1}} &\leqslant C(g)\varepsilon_j^2 \\ &\to 0.\end{aligned}$$

The localized estimate of Guan and Wang [12] implies that for any $\rho > 1$, there is a constant $C = C(\rho, g)$ such that

$$\max_{B(0,\rho)} |\nabla \tilde{u}_j|^2 \leqslant C. \tag{3.5}$$

Combining this gradient bound with the condition $\tilde{u}_j(0) = 0$ we see that

$$\max_{B(0,\rho)} (|\tilde{u}_j| + |\nabla \tilde{u}_j|) \leqslant C(\rho), \tag{3.6}$$

for any $\rho > 1$. With this estimate we can appeal to the local C^2-estimates of Guan and Wang [12, Proposition 3]. Once again, our equation is slightly different, but this time (in light of the C^1-estimates for $\tilde{u}_j$) the required modifications are minor. We will omit the details. As a consequence, on any ball $B(0, \rho)$, $\tilde{u}_j$ satisfies

$$\max_{B(0,\rho)} (|\tilde{u}_j| + |\nabla \tilde{u}_j| + |\nabla^2 \tilde{u}_j|) \leqslant C(\rho). \tag{3.7}$$

It follows from the work of Evans and Krylov that one obtains $C^{2,\alpha}$-estimates for $\tilde{u}_j$ on any fixed ball, and consequently $\{\tilde{u}_j\}$ converges uniformly in the $C^{2,\alpha}$-topology on compact sets to a solution u of

$$\sigma_k^{1/k}\left(\nabla^2 u + du \otimes du - \tfrac{1}{2}|\nabla u|^2 g\right) = \sigma_k^{1/k}(S^n)e^{-2u}. \tag{3.8}$$

The aforementioned regularity results imply that $u \in C^\infty$.

By the classification result of Li and Li [19], all solutions of (3.8) are obtained by pulling back the round metric on the sphere (and its images under conformal diffeomorphisms) via stereographic projection. In particular,

$$vol(e^{-2u}ds^2) = vol(S^n). \tag{3.9}$$

Lemma 3.1.

$$\liminf_j \ vol(e^{-2u_j}g) \geqslant vol(S^n). \tag{3.10}$$

Proof. Given $\eta > 0$, fix a large ball $B = B(0, \rho_0) \subset \mathbf{R}^n$ such that

$$\int_B e^{-n\tilde{u}_j}\, dvol_{g_j} > vol(S^n) - \eta \tag{3.11}$$

for all $j \geqslant J$. Pulling back to M^n by T_j^{-1}, we have

$$\begin{aligned} vol(e^{-2u_j}g) = \int e^{-nu_j}\, dvol_g \geqslant & \int_{B(z_j, \varepsilon_j \rho_0)} e^{-nu_j}\, dvol_g \\ &= \int_B e^{-n\tilde{u}_j}\, dvol_{g_j} \\ &> vol(S^n) - \eta. \end{aligned} \tag{3.12}$$

This proves the Lemma. $\square$

By Eq. (2.12), for $t \geqslant \frac{1}{2}$, u_j satisfies

$$\sigma_k^{1/k}\left(A + \nabla^2 u_j + du_j \otimes du_j - \tfrac{1}{2}|\nabla u_j|^2 g\right) \geqslant \sigma_k^{1/k}(S^n)e^{-2u_j}. \tag{3.13}$$

Therefore, $g_j = e^{-2u_j}g$ satisfies

$$\sigma_k^{1/k}(g_j^{-1}A_{u_j}) \geqslant \sigma_k^{1/k}(S^n). \tag{3.14}$$

From Lemma 3.1 we conclude $\Lambda_k(M^n, [g]) = vol(S^n)$, which is a contradiction. Therefore, the sequence $\{u_j\}$ is bounded below, as claimed. $\square$

To complete the proof of the theorem, we may argue exactly as in the proof of Theorem 2.1. Namely, the localized C^1-estimate of Guan and Wang together with the lower bound (3.2) implies a gradient bound for u, and consequently the Harnack inequality (2.29). We may then apply Lemma 2.5 to conclude that u has an a priori upper bound. Higher order estimates follow, just as we described at the end of the proof of Theorem 2.1. $\square$

The preceding blow-up argument can be applied, with only minor modifications, to prove the compactness of solutions of (1.6). The details will be omitted.

To establish existence, we apply the degree theory for fully nonlinear equations as developed in [21]. In Section 2 we showed that the Leray–Schauder degree of a solution of (2.12) at $t = 0$ is nonzero. We remark that Eq. (2.12) differs from that considered in [21] only by the presence of the integral term. From the compactness established in Theorem 2.1, this integral term is bounded. Furthermore, the proof in [21] relies on differentiating the equation. Since the integral term is a constant, the definition of degree and proof of invariance of degree under homotopy are valid for Eq. (2.12). We conclude that the Leray–Schauder degree at $t = 1$ (with respect to a sufficiently large ball in $C^{4,\alpha}$) is nonzero, and consequently there exists a solution at $t = 1$.

4. Sharp estimates for Λ_k

In this section we prove various estimates for the conformal invariant Λ_k. We begin by describing some general properties which are independent of the dimension, then consider the cases $n = 3$ and 4 separately.

A basic tool in many of our results is the *Newton–Maclaurin inequality* (see [15]): if $(\lambda_1, \dots, \lambda_n) \in \Gamma_k^+$ and $k \geqslant j$, then

$$\binom{n}{k}^{-1/k} \sigma_k^{1/k}(\lambda_1, \dots, \lambda_n) \leqslant \binom{n}{j}^{-1/j} \sigma_j^{1/j}(\lambda_1, \dots, \lambda_n).$$

This implies

Lemma 4.1. *If $g \in \Gamma_k^+(M^n)$ and $j \leqslant k$, then*

$$\Lambda_j(M^n, [g]) \geqslant \Lambda_k(M^n, [g]). \tag{4.1}$$

As a consequence of this Lemma, in order to estimate Λ_k with $k > n/2$ it typically suffices to estimate Λ_j, where $j = \left[\frac{n}{2}\right] + 1$.

The proof of Proposition 1.1. The proof is based on the sharp inequality of Guan et al. [10]: If $g \in \Gamma_k^+(M^n)$ with $k > n/2$, then the Ricci tensor satisfies

$$Ric \geqslant \frac{(2k-n)}{2n(k-1)} Rg, \tag{4.2}$$

where R is the scalar curvature of g. The finiteness of $\Lambda_k(M^n, [g])$ will follow from a lower bound for the scalar curvature and the Bishop Comparison Theorem, as we now explain.

First, by the Newton–MacLaurin inequality

$$\sigma_k^{1/k}(g^{-1}A) \leqslant \frac{1}{n}\binom{n}{k}^{1/k} \sigma_1(g^{-1}A) = c(k,n)R. \tag{4.3}$$

If g satisfies

$$\sigma_k^{1/k}(g^{-1}A) \geqslant \sigma_k^{1/k}(S^n), \tag{4.4}$$

then combining (4.3) and (4.4) we have

$$R \geqslant c(k,n)^{-1}\sigma_k^{1/k}(S^n) > 0.$$

Substituting this into (4.2) gives

$$Ric \geqslant \frac{(2k-n)}{2n(k-1)} c(k,n)^{-1}\sigma_k^{1/k}(S^n)g.$$

Since $n/2 < k \leqslant n$, we obtain a lower bound for Ric which only depends on the dimension:

$$Ric \geqslant c(n)g.$$

By Myer's theorem, the diameter of g is bounded by a constant $C = C(n)$:

$$diam(M^n, g) \leqslant C.$$

In addition, by the Bishop comparison theorem the positivity of the Ricci curvature implies the volume of a geodesic ball of radius ρ in g is bounded by $vol(B(\rho)) \leqslant C_n\rho^n$. This fact, combined with the diameter estimate above, implies that $vol(M^n, g) \leqslant C(n)$. Thus,

$$\Lambda_k(M^n, [g]) \leqslant C(n).$$

This completes the proof. $\square$

4.1. $n = 3$

We now turn to three dimensions, where the sharp estimates of Λ_k are based on the following result of Bray [3]:

Theorem 4.1 (Bray's Football Theorem). *Let (S^3, g_0) be the constant curvature metric on S^3 with scalar curvature R_0, Ricci tensor $Ric_0\, g_0$, and volume V_0. If $\varepsilon \in (0,1]$ and (M^3, g) is any complete smooth Riemannian manifold of volume V*

satisfying

$$R(g) \geqslant R_0, \tag{4.5}$$

$$Ric(g) \geqslant \varepsilon \, Ric_0 \, g, \tag{4.6}$$

then

$$V \leqslant \alpha(\varepsilon) \, V_0, \tag{4.7}$$

where

$$\alpha(\varepsilon) = \sup_{\frac{4\pi}{3-2\varepsilon} \leqslant z \leqslant 4\pi} \frac{1}{\pi^2} \begin{pmatrix} \int_0^{y(z)} (36\pi - 27(1-\varepsilon) y(z)^{\frac{2}{3}} - 9\varepsilon x^{\frac{2}{3}})^{-\frac{1}{2}} \, dx \\ + \int_{y(z)}^{z^{\frac{3}{2}}} (36\pi - 18(1-\varepsilon) y(z) x^{-\frac{1}{3}} - 9 x^{\frac{2}{3}})^{-\frac{1}{2}} \, dx \end{pmatrix},$$

where

$$y(z) = \frac{z^{\frac{1}{2}}(4\pi - z)}{2(1-\varepsilon)}.$$

Furthermore, *this expression for* $\alpha(\varepsilon)$ *is sharp*.

When $\varepsilon = 1$, the lower bound on the scalar curvature (4.5) follows from the lower bound on the Ricci curvature (4.6), and the result is equivalent to Bishop's inequality. Now define

$$\varepsilon_0 = \inf\{\varepsilon \in (0,1] \mid \alpha(\varepsilon) = 1\}.$$

Bray's theorem is remarkable precisely because $\varepsilon_0 < 1$. Although Bray claimed this fact in his thesis, he did not include the proof. However, he did provide compelling numerical evidence suggesting $\varepsilon_0 = 0.134\ldots$. This value of ε_0 corresponds to a rotationally symmetric manifold resembling a football; thus the name. In any case, there are currently no rigorous estimates of ε_0 from above.

For our purposes we need to know that $\varepsilon_0 \leqslant 0.5$. To see why, suppose $g \in \Gamma_2^+(M^3)$ satisfies

$$\sigma_2^{1/2}(g^{-1}A) \geqslant \sigma_2^{1/2}(g_0^{-1}A_0). \tag{4.8}$$

Then the Newton–Maclaurin inequality implies

$$R \geqslant R_0. \tag{4.9}$$

In addition, by inequality (4.2),

$$\begin{aligned} Ric(g) &\geqslant \frac{1}{6} Rg \\ &\geqslant \frac{1}{6} R_0 g \\ &= \frac{1}{2} Ric_0\, g. \end{aligned} \tag{4.10}$$

Therefore, if $\varepsilon_0 \leqslant \frac{1}{2}$, from Bray's theorem we would conclude

$$\sigma_2^{1/2}(g^{-1}A) \geqslant \sigma_2^{1/2}(g_0^{-1}A_0) \quad \Rightarrow \quad vol(M^3, g) \leqslant vol(S^3). \tag{4.11}$$

Consequently,

$$\Lambda_2(M^3, [g]) \leqslant vol(S^3),$$

and Theorem 1.2 would follow.

A similar argument can be used to prove inequality (1.12), again provided $\varepsilon_0 \leqslant \frac{1}{2}$. Under the assumptions of Theorem 1.3, we know from inequality (4.2) that g has positive Ricci curvature. By Meyer's theorem the fundamental group of M^3 is finite. If we let $\widetilde{M}^3$ denote the universal cover of M^3, then $\widetilde{M}^3$ is compact and the volume of M^3 and $\widetilde{M}^3$ are related by

$$vol(\widetilde{M}^3, \tilde{g}) = ||\pi_1(M^3)|| vol(M^3, g), \tag{4.12}$$

where $\tilde{g}$ denotes the lift of g to $\widetilde{M}^3$. Applying the volume estimate (4.11) to the cover $(\widetilde{M}^3, \tilde{g})$ and using (4.12), we arrive at (1.12). A similar argument can be used to prove Theorem 1.6.

The main result of this subsection is a rigorous proof of the inequality $\varepsilon_0 \leqslant \frac{1}{2}$. Before providing the details of this estimate, however, it may be helpful to sketch an outline of Bray's proof.

Given a real number $V \geqslant 0$, define

$$A(V) = \inf_{\Omega} \{area(\partial\Omega) \mid vol(\Omega) = V\}, \tag{4.13}$$

where Ω is any region in M^3, $vol(\Omega)$ is the volume of Ω, and $area(\partial\Omega)$ is the two-dimensional surface area of the boundary. Since M^3 is compact, there always exists a smooth region whose boundary $\Sigma(V)$ attains the infimum $A(V)$. Of course, $\Sigma(V)$ will necessarily have constant mean curvature.

For a fixed value $V = V_0$ we consider a normal variation of $\Sigma(V_0)$, parametrized by the volume V. Let $A_{V_0}(V)$ denote the area of this variation, and primes denote differentiation with respect to V. Then $A'_{V_0}(V) = H$, where H is the mean curvature of $\Sigma(V_0)$, and

$$A_{V_0}(V_0)^2 A''_{V_0}(V_0) = \int_{\Sigma(V_0)} [-||\Pi||^2 - Ric(\nu,\nu)], \tag{4.14}$$

where Π is the second fundamental form of $\Sigma(V_0)$ and ν is a unit normal. From inequality (4.6) and

$$||\Pi||^2 \geqslant \tfrac{1}{2} H^2, \tag{4.15}$$

we conclude

$$A''_{V_0}(V_0) \leqslant -\frac{1}{A_{V_0}(V_0)}\left(\frac{1}{2} A'_{V_0}(V_0)^2 + \varepsilon\, Ric_0\right). \tag{4.16}$$

Since $\Sigma_{V_0}(V)$ may not attain the infimum in (4.13), $A(V) \leqslant A_{V_0}(V)$. Thus, $A(V)$ satisfies

$$A''(V) \leqslant -\frac{1}{A(V)}\left(\frac{1}{2} A'(V)^2 + \varepsilon\, Ric_0\right). \tag{4.17}$$

By the Gauss equation,

$$Ric(\nu,\nu) = \tfrac{1}{2} R - K + \tfrac{1}{2} H^2 - \tfrac{1}{2} ||\Pi||^2,$$

where K is the Gauss curvature of $\Sigma(V_0)$. Substituting this into (4.14) gives

$$A_{V_0}(V_0)^2 A''_{V_0}(V_0) = \int_{\Sigma(V_0)} \left[-\frac{1}{2} R + K - \frac{1}{2} H^2 + \frac{1}{2} ||\Pi||^2\right]. \tag{4.18}$$

As Bray points out, the positivity of the Ricci curvature implies that $\Sigma(V_0)$ is connected. Thus, applying the Gauss–Bonnet formula and appealing to inequalities (4.9) and (4.15) we get

$$A''_{V_0}(V_0) \leqslant \frac{4\pi}{A_{V_0}(V_0)^2} - \frac{1}{A_{V_0}(V_0)}\left(\frac{3}{4} A'_{V_0}(V_0)^2 + \frac{1}{2} R_0\right). \tag{4.19}$$

As before, since $A(V) \leqslant A_{V_0}(V)$ we have

$$A''(V) \leqslant \frac{4\pi}{A(V)^2} - \frac{1}{A(V)}\left(\frac{3}{4} A'(V)^2 + \frac{1}{2} R_0\right). \tag{4.20}$$

Next, let

$$F(V) = A(V)^{3/2}. \tag{4.21}$$

By (4.17) and (4.20), F satisfies

$$F''(V) \leqslant -\frac{3\varepsilon}{2} Ric_0\, F(V)^{-\frac{1}{3}}, \tag{4.22}$$

$$F''(V) \leqslant \frac{36\pi - F'(V)^2}{6F(V)} - \frac{3}{4} R_0 F(V)^{-\frac{1}{3}}. \tag{4.23}$$

Of course, one needs to properly interpret the sense in which these inequalities hold; see [3] for precise notions.

Combining (4.22) and (4.23), we have

$$F''(V) \leqslant -\frac{1}{2} F^{-\frac{1}{3}} \max\left\{ -\frac{36\pi - F'(V)^2}{3F(V)^{\frac{2}{3}}} + \frac{3}{2} R_0, 3\varepsilon\, Ric_0 \right\}. \tag{4.24}$$

Consider the phase space associated to this differential inequality, which we view as the xy-plane with $x = F(V)$ and $y = F'(V)$. Let γ be a path in phase space with initial value $V = 0$ and terminal value $V = \frac{1}{2} vol(M^3, g)$. Then γ starts at a point on the (positive) y-axis and ends at a point on the (positive) x-axis. By (4.24) this path must satisfy the differential inequality

$$\frac{dy}{dx} \leqslant -\frac{1}{2} x^{-\frac{1}{3}} y^{-1} \max\left\{ -\frac{(36\pi - y^2)}{3x^{\frac{2}{3}}} + \frac{3}{2} R_0, 3\varepsilon\, Ric_0 \right\}. \tag{4.25}$$

Also,

$$\frac{1}{2} vol(M^3, g) = \int_\gamma dV = \int_\gamma \frac{dx}{y}. \tag{4.26}$$

A path which maximizes the line integral (4.26) will be a path which attains equality in (4.25). This results in an ODE which can be explicitly solved, and by evaluating the line integral for this path one obtains an upper estimate on the volume as in (4.7).

With this brief overview in mind, we now give an estimate of ε_0.

Theorem 4.2. *The constant* $\varepsilon_0 \leqslant \frac{1}{2}$.

Proof. According to Bray's theorem, it suffices to show that $\alpha\left(\frac{1}{2}\right) = 1$; i.e., that

$$\sup_{2\pi \leqslant z \leqslant 4\pi} \frac{1}{\pi^2} \begin{pmatrix} \int_0^{y(z)} \left(36\pi - \frac{27}{2} y(z)^{\frac{2}{3}} - \frac{9}{2} x^{\frac{2}{3}}\right)^{-\frac{1}{2}} dx \\ + \int_{y(z)}^{z^{\frac{3}{2}}} (36\pi - 9y(z)x^{-\frac{1}{3}} - 9x^{\frac{2}{3}})^{-\frac{1}{2}}\, dx \end{pmatrix} = 1, \tag{4.27}$$

where

$$y = y(z) = z^{\frac{1}{2}}(4\pi - z). \tag{4.28}$$

To this end, let

$$I_1(z) = \frac{1}{\pi^2}\int_0^y \left(36\pi - \frac{27}{2}y^{\frac{2}{3}} - \frac{9}{2}x^{\frac{2}{3}}\right)^{-\frac{1}{2}} dx, \tag{4.29}$$

$$I_2(z) = \frac{1}{\pi^2}\int_y^{z^{\frac{3}{2}}} \left(36\pi - 9yx^{-\frac{1}{3}} - 9x^{\frac{2}{3}}\right)^{-\frac{1}{2}} dx. \tag{4.30}$$

We want to show that for $z \in [2\pi, 4\pi]$,

$$I_1(z) + I_2(z) \leqslant 1. \tag{4.31}$$

The first integral in (4.31) can be evaluated in closed form. The second integral can be expressed in terms of elliptic functions, though the resulting formula seems difficult to estimate. Instead of this approach, we will perform a change of variable and approximate the new integrand by one which can also be evaluated in closed form. It turns out to be much easier estimating both integrals in terms of this new variable; for this reason we begin by analyzing I_2, where the substitution originates.

Let $x = t^3$; then

$$\begin{aligned} I_2 &= \frac{1}{\pi^2}\int_{y^{\frac{1}{3}}}^{z^{\frac{1}{2}}} [36\pi - 9yt^{-1} - 9t^2]^{-\frac{1}{2}}(3t^2)\, dt \\ &= \frac{1}{\pi^2}\int_{y^{\frac{1}{3}}}^{z^{\frac{1}{2}}} \frac{t^{\frac{5}{2}}\, dt}{\sqrt{4\pi t - y - t^3}}. \end{aligned} \tag{4.32}$$

Note that the denominator factors:

$$4\pi t - y - t^3 = (z^{\frac{1}{2}} - t)(t^2 + z^{\frac{1}{2}}t - (4\pi - z)).$$

Therefore,

$$I_2 = \frac{1}{\pi^2}\int_{y^{\frac{1}{3}}}^{z^{\frac{1}{2}}} \frac{t^{\frac{5}{2}}\, dt}{\sqrt{(z^{\frac{1}{2}} - t)(t^2 + z^{\frac{1}{2}}t - (4\pi - z))}}.$$

Now perform another change of variable: let $s = tz^{-\frac{1}{2}}$; then

$$I_2 = \frac{z}{\pi^2}\int_{y^{\frac{1}{3}}z^{-\frac{1}{2}}}^{1} \frac{s^{\frac{5}{2}}\,ds}{\sqrt{(1-s)\left(s^2+s-\left(\frac{4\pi-z}{z}\right)\right)}}.$$

Let $\varphi = y^{\frac{1}{3}}z^{-\frac{1}{2}}$. By (4.28),

$$\varphi^3 = \left(\frac{4\pi - z}{z}\right), \tag{4.33}$$

$$z = \frac{4\pi}{1+\varphi^3}. \tag{4.34}$$

Therefore,

$$I_2 = \left(\frac{4}{\pi}\right)\left(\frac{1}{1+\varphi^3}\right)\int_{\varphi}^{1} \frac{s^{\frac{5}{2}}\,ds}{\sqrt{(1-s)(s^2+s-\varphi^3)}}. \tag{4.35}$$

Since z is a decreasing function of φ, we can change variables and view I_1 and I_2 as functions of φ (instead of z). Note that $2\pi \leqslant z \leqslant 4\pi$, while $0 \leqslant \varphi \leqslant 1$.

By doing a simple substitution the first integral can be evaluated in closed form:

$$\begin{aligned} I_1(z) &= \frac{1}{\pi^2}\int_0^y \left(36\pi - \frac{27}{2}y^{\frac{2}{3}} - \frac{9}{2}x^{\frac{2}{3}}\right)^{-\frac{1}{2}} dx \\ &= \frac{\sqrt{2}}{2\pi^2}(8\pi - 3y^{\frac{2}{3}})\left[\arcsin\left(\frac{y^{\frac{1}{3}}}{\sqrt{8\pi - 3y^{\frac{2}{3}}}}\right) - \frac{2y^{\frac{1}{3}}\sqrt{2\pi - y^{\frac{2}{3}}}}{8\pi - 3y^{\frac{2}{3}}}\right]. \end{aligned} \tag{4.36}$$

In order to rewrite (4.36) in terms of φ, we need to first express y in terms of φ. By (4.28) and (4.34),

$$y = \frac{(4\pi)^{\frac{3}{2}}\varphi^3}{(1+\varphi^3)^{\frac{3}{2}}}. \tag{4.37}$$

Substituting this into (4.36) and carrying out the obvious simplifications, the result is

$$\begin{aligned} I_1(\varphi) = \left(\frac{4}{\pi}\right)\left(\frac{1}{1+\varphi^3}\right)\left(\frac{\sqrt{2}}{2}\right)\Bigg[&(2+2\varphi^3-3\varphi^2)\arcsin\left(\frac{\varphi}{(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}}}\right) \\ &- (2+2\varphi^3-4\varphi^2)^{\frac{1}{2}}\varphi\Bigg]. \end{aligned} \tag{4.38}$$

To establish the inequality

$$I_1(\varphi) + I_2(\varphi) \leqslant 1 \quad \text{for } \varphi \in [0,1] \tag{4.39}$$

we divide the interval $[0,1]$ into two parts. This division, or something like it, seems necessary, since the contribution of the two integrals in the sum above is different for φ near 0 and φ near 1. More precisely, $I_1(\varphi) \to 0$ and $I_2(\varphi) \to 1$ as $\varphi \to 0$, while $I_1(\varphi) \to 1/\sqrt{2}$ and $I_2(\varphi) \to 0$ as $\varphi \to 1$. Therefore, in the subsections which follow we derive our estimates first on the interval $\left[0, \frac{4}{5}\right]$, then on $\left[\frac{4}{5}, 1\right]$.

4.1.1. Estimate from 0 to $\frac{4}{5}$

We begin with an estimate of I_1:

Proposition 4.1. *For $\varphi \in \left[0, \frac{4}{5}\right]$,*

$$I_1 \leqslant \left(\frac{4}{\pi}\right)\left(\frac{1}{1+\varphi^3}\right)\left(\frac{61}{100}\varphi^3\right).$$

Proof. The proof relies on a sharp estimate of the arcsin term in (4.38).

Lemma 4.2. *If $\beta \in (0,1]$, then for $x \in [0,\beta]$*

$$\arcsin x \leqslant x + mx^3, \tag{4.40}$$

where

$$m = \left(\frac{\arcsin \beta - \beta}{\beta^3}\right).$$

Proof. This is equivalent to the inequality

$$\theta \leqslant \sin\theta + m\sin^3\theta, \quad \theta \in [0, \arcsin\beta].$$

Let $f(\theta) = \sin\theta + m\sin^3\theta - \theta$. We want to see that $f(\theta) \geqslant 0$ for $\theta \in [0, \arcsin\beta]$. Note that $f(0) = 0, f(\arcsin\beta) = 0$. Thus, to show that $f(\theta) \geqslant 0$ it suffices to show that: (i) $f(\theta) > 0$ for $\theta > 0$ small, and (ii) f' has exactly one zero in the open interval $(0, \arcsin\beta)$. Of course, since $f(0) = f(\arcsin\beta) = 0$ Rolle's theorem guarantees that $f'(\theta_0) = 0$ for some $\theta_0 \in (0, \arcsin\beta)$.

If we write out the Taylor expansion of f near zero,

$$f(\theta) = \left(m - \tfrac{1}{6}\right)\theta^3 + O(\theta^5).$$

Thus, if we can show that $m > \frac{1}{6}$ then (i) will follow. To this end, define another function $h(\beta) = \arcsin \beta - \beta - \frac{1}{6}\beta^3$. Then

$$h'(\beta) = \frac{1}{\sqrt{1-\beta^2}} - 1 - \frac{1}{2}\beta^2.$$

It is easy to see that $h'(\beta) > 0$ for $\beta \in (0,1)$: just differentiate again and use the fact that $h'(0) = 0$. Thus, $h(\beta) > h(0) = 0$ for $\beta \in (0,1)$, which implies $m > \frac{1}{6}$.

To prove (ii), note $f'(\theta) = (1+3m)\cos\theta - 3m\cos^3\theta - 1$. Let

$$p(z) = (1+3m)z - 3mz^3 - 1.$$

If we can show that p has exactly one zero in the interval $(\cos(\arcsin\beta), 1) = \left(\sqrt{1-\beta^2}, 1\right)$, then (ii) will follow. Let $z_0 = \cos\theta_0$; then $p(z_0) = 0$. Also, $p(1) = 0$. Thus, p has two zeros in the closed interval $\left[\sqrt{1-\beta^2}, 1\right]$: $z_0 \in \left(\sqrt{1-\beta^2}, 1\right)$, and $z_1 = 1$. Since p is a cubic polynomial, it must have a third zero z_2. But notice

$$\lim_{z\to-\infty} p(z) = +\infty$$

while $p(0) = -1$. Consequently, $z_2 < 0$, and p has only one zero in the open interval $\left(\sqrt{1-\beta^2}, 1\right)$. $\square$

Using the preceding Lemma, we estimate the arcsin term in (4.38) as follows. First, observe that

$$2 + 2\varphi^3 - 3\varphi^2 \geqslant 1. \tag{4.41}$$

This follows from the fact that $\varphi^2 \leqslant \frac{2}{3}\varphi^3 + \frac{1}{3}$, and hence $-3\varphi^2 \geqslant -2\varphi^3 - 1$. A consequence of (4.41) is that

$$\frac{\varphi}{(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}}} \leqslant \varphi.$$

Therefore, $x \equiv \varphi/(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}} \in \left[0, \frac{4}{5}\right]$ whenever $\varphi \in \left[0, \frac{4}{5}\right]$. From (4.40) we conclude

$$\arcsin\left(\frac{\varphi}{(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}}}\right) \leqslant \frac{\varphi}{(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}}} + m_0 \frac{\varphi^3}{(2+2\varphi^3-3\varphi^2)^{\frac{3}{2}}},$$

where

$$m_0 = \left(\frac{\arcsin\frac{4}{5} - \frac{4}{5}}{\left(\frac{4}{5}\right)^3}\right). \tag{4.42}$$

Therefore,

$$\begin{aligned}
&(2+2\varphi^3-3\varphi^2)\arcsin\left(\frac{\varphi}{(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}}}\right) - (2+2\varphi^3-4\varphi^2)^{\frac{1}{2}}\varphi \\
&\quad\leqslant (2+2\varphi^3-3\varphi^2)^{\frac{1}{2}}\varphi + m_0\frac{\varphi^3}{(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}}} - (2+2\varphi^3-4\varphi^2)^{\frac{1}{2}}\varphi \\
&\quad= [(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}} - (2+2\varphi^3-4\varphi^2)^{\frac{1}{2}}]\varphi + m_0\frac{\varphi^3}{(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}}}
\end{aligned}$$

whenever $\varphi \in \left[0, \frac{4}{5}\right]$. For the first term above,

$$\begin{aligned}
&(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}} - (2+2\varphi^3-4\varphi^2)^{\frac{1}{2}} \\
&\quad= \frac{[(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}} - (2+2\varphi^3-4\varphi^2)^{\frac{1}{2}}][(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}} + (2+2\varphi^3-4\varphi^2)^{\frac{1}{2}}]}{[(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}} + (2+2\varphi^3-4\varphi^2)^{\frac{1}{2}}]} \\
&\quad= \frac{\varphi^2}{[(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}} + (2+2\varphi^3-4\varphi^2)^{\frac{1}{2}}]}.
\end{aligned}$$

On the interval $\left[0, \frac{4}{5}\right]$, $l(\varphi) = \frac{(2+2\varphi^3-4\varphi^2)}{(2+2\varphi^3-3\varphi^2)}$ is decreasing; thus $(2+2\varphi^3-4\varphi^2)^{\frac{1}{2}} \geqslant l\left(\frac{4}{5}\right)^{\frac{1}{2}}(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}}$. Substituting this into the expression above,

$$(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}} - (2+2\varphi^3-4\varphi^2)^{\frac{1}{2}} \leqslant \frac{\varphi^2}{\left(1 + l\left(\frac{4}{5}\right)^{\frac{1}{2}}\right)(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}}}.$$

Therefore,

$$\begin{aligned}
&(2+2\varphi^3-3\varphi^2)\arcsin\left(\frac{\varphi}{(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}}}\right) - (2+2\varphi^3-4\varphi^2)^{\frac{1}{2}}\varphi \\
&\quad\leqslant \left(\frac{1}{1 + l\left(\frac{4}{5}\right)^{\frac{1}{2}}} + m_0\right)\frac{\varphi^3}{(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}}}.
\end{aligned}$$

Substituting this into (4.38) we conclude

$$
\begin{aligned}
I_1(\varphi) &\leqslant \left(\frac{4}{\pi}\right)\left(\frac{1}{1+\varphi^3}\right)\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{1+l\left(\frac{4}{5}\right)^{\frac{1}{2}}}+m_0\right)\varphi^3 \\
&= \left(\frac{4}{\pi}\right)\left(\frac{1}{1+\varphi^3}\right)c_0\varphi^3,
\end{aligned}
$$

where

$$
\begin{aligned}
c_0 &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{1+l\left(\frac{4}{5}\right)^{\frac{1}{2}}}+m_0\right) \\
&= \left(\frac{\sqrt{2}}{2}\right)\left[\frac{1}{1+\left(\frac{29}{69}\right)^{\frac{1}{2}}}+\left(\frac{\arcsin\frac{4}{5}-\frac{4}{5}}{\left(\frac{4}{5}\right)^3}\right)\right] \\
&= 0.604795\ldots \\
&< \frac{61}{100}. \qquad \square
\end{aligned}
$$

To estimate I_2, we begin by rewriting the integrand in (4.35):

$$
\begin{aligned}
\frac{s^{\frac{5}{2}}}{\sqrt{(1-s)(s^2+s-\varphi^3)}} &= \frac{s^{\frac{5}{2}}}{\sqrt{(1-s)(s^2+s)}}\sqrt{\frac{s^2+s}{s^2+s-\varphi^3}} \\
&= \frac{s^2}{\sqrt{(1-s)(1+s)}}f(s),
\end{aligned}
\tag{4.43}
$$

where

$$
f(s) = \frac{(s^2+s)^{\frac{1}{2}}}{(s^2+s-\varphi^3)^{\frac{1}{2}}}.
$$

Differentiating,

$$
f'(s) = -\tfrac{1}{2}\varphi^3(2s+1)(s^2+s-\varphi^3)^{-\frac{3}{2}}(s^2+s)^{-\frac{1}{2}}.
$$

Since

$$
s^2+s-\varphi^3 \leqslant s^2+s,
$$

it follows

$$
(s^2+s-\varphi^3)^{-\frac{3}{2}} \geqslant (s^2+s)^{-\frac{3}{2}}.
$$

Therefore,

$$f'(s) \leqslant -\frac{1}{2}\varphi^3 \frac{(2s+1)}{(s^2+s)^2}.$$

By the fundamental theorem of calculus,

$$\begin{aligned} f(s) - f(\varphi) &\leqslant \int_{\varphi}^{s} -\frac{1}{2}\varphi^3 \frac{(2x+1)}{(x^2+x)^2}\,dx \\ &= \frac{1}{2}\varphi^3 (x^2+x)^{-1}\Big|_{x=\varphi}^{x=s} \\ &= \frac{1}{2}\varphi^3\left(\frac{1}{s^2+s} - \frac{1}{\varphi^2+\varphi}\right), \end{aligned}$$

hence

$$f(s) \leqslant \left(f(\varphi) - \frac{1}{2}\frac{\varphi^2}{1+\varphi}\right) + \frac{1}{2}\varphi^3 \frac{1}{s(1+s)}.$$

Substituting this inequality into (4.43) we have

$$\begin{aligned} &\int_{\varphi}^{1} \frac{s^2}{\sqrt{(1-s)(1+s)}} f(s)\,ds \\ &\leqslant \int_{\varphi}^{1} \frac{s^2}{\sqrt{(1-s)(1+s)}}\left[\left(f(\varphi) - \frac{1}{2}\frac{\varphi^2}{1+\varphi}\right) + \frac{1}{2}\varphi^3 \frac{1}{s(1+s)}\right] ds \\ &= \left(f(\varphi) - \frac{1}{2}\frac{\varphi^2}{1+\varphi}\right)\int_{\varphi}^{1} \frac{s^2}{\sqrt{1-s^2}}\,ds + \frac{1}{2}\varphi^3 \int_{\varphi}^{1} \frac{s}{\sqrt{(1-s)(1+s)^3}}\,ds. \end{aligned} \tag{4.44}$$

Both integrals in (4.44) are elementary:

$$\begin{aligned} &\left(f(\varphi) - \frac{1}{2}\frac{\varphi^2}{1+\varphi}\right)\int_{\varphi}^{1} \frac{s^2}{\sqrt{1-s^2}}\,ds \\ &\quad = \left(f(\varphi) - \frac{1}{2}\frac{\varphi^2}{1+\varphi}\right)\left[-\frac{1}{2}s\sqrt{1-s^2} + \frac{1}{2}\arcsin s\right]_{s=\varphi}^{s=1} \\ &\quad = \left(f(\varphi) - \frac{1}{2}\frac{\varphi^2}{1+\varphi}\right)\left[\frac{\pi}{4} + \frac{1}{2}\varphi\sqrt{1-\varphi^2} - \frac{1}{2}\arcsin\varphi\right], \end{aligned}$$

$$\begin{aligned} \frac{1}{2}\varphi^3 \int_{\varphi}^{1} \frac{s}{\sqrt{(1-s)(1+s)^3}}\,ds &= \frac{1}{2}\varphi^3\left[\sqrt{\frac{1-s}{1+s}} + \arcsin s\right]_{s=\varphi}^{s=1} \\ &= \frac{1}{2}\varphi^3\left[\frac{\pi}{2} - \sqrt{\frac{1-\varphi}{1+\varphi}} - \arcsin\varphi\right]. \end{aligned}$$

Combining the above and substituting into (4.35) we get

$$
\begin{aligned}
I_2 \leqslant & \left(\frac{4}{\pi}\right)\left(\frac{1}{1+\varphi^3}\right)\left\{\left(\frac{(1+\varphi)^{\frac{1}{2}}}{(1+\varphi-\varphi^2)^{\frac{1}{2}}}-\frac{1}{2}\frac{\varphi^2}{1+\varphi}\right)\left[\frac{\pi}{4}+\frac{1}{2}\varphi\sqrt{1-\varphi^2}-\frac{1}{2}\arcsin\varphi\right]\right.\\
& \left. +\ \varphi^3\left[\frac{\pi}{2}-\sqrt{\frac{1-\varphi}{1+\varphi}}-\arcsin\varphi\right]\right\}\\
= & \left(\frac{4}{\pi}\right)\left(\frac{1}{1+\varphi^3}\right)\{E(\varphi)F(\varphi)+G(\varphi)\varphi^3\},
\end{aligned} \tag{4.45}
$$

where

$$
E(\varphi)=\frac{(1+\varphi)^{\frac{1}{2}}}{(1+\varphi-\varphi^2)^{\frac{1}{2}}}-\frac{1}{2}\frac{\varphi^2}{1+\varphi}, \tag{4.46}
$$

$$
F(\varphi)=\frac{\pi}{4}+\frac{1}{2}\varphi\sqrt{1-\varphi^2}-\frac{1}{2}\arcsin\varphi, \tag{4.47}
$$

$$
G(\varphi)=\frac{\pi}{4}-\frac{1}{2}\sqrt{\frac{1-\varphi}{1+\varphi}}-\frac{1}{2}\arcsin\varphi. \tag{4.48}
$$

Proposition 4.2. *For* $\varphi \in \left[0, \frac{4}{5}\right]$,

$$
I_2 \leqslant \left(\frac{4}{\pi}\right)\left(\frac{1}{1+\varphi^3}\right)\left\{\frac{\pi}{4}+\left(\frac{\pi}{16}-\frac{1}{3}\right)\varphi^3+\left(\frac{\pi}{4}-\frac{1}{2}\right)\varphi^3\right\}. \tag{4.49}
$$

Proof. The proof of (4.49) is based on the following estimates of E, F, and G.

Lemma 4.3. (i) *For* $\varphi \in \left[0, \frac{2}{5}\right]$,

$$
E(\varphi) \leqslant 1+\tfrac{1}{2}\varphi^4. \tag{4.50}
$$

(ii) *For* $\varphi \in \left[\frac{2}{5}, \frac{4}{5}\right]$,

$$
E(\varphi) \leqslant 1+\frac{125}{434}\varphi^4. \tag{4.51}
$$

Proof. First, write

$$
E(\varphi)=E_1(\varphi)+E_2(\varphi),
$$

where

$$E_1(\varphi) = \frac{(1+\varphi)^{\frac{1}{2}}}{(1+\varphi-\varphi^2)^{\frac{1}{2}}},$$

$$E_2(\varphi) = -\frac{1}{2}\frac{\varphi^2}{1+\varphi}.$$

Then

$$\begin{aligned} E_1(\varphi) &= \frac{(1+\varphi)^{\frac{1}{2}}}{(1+\varphi-\varphi^2)^{\frac{1}{2}}} - 1 + 1 \\ &= \frac{(1+\varphi)^{\frac{1}{2}} - (1+\varphi-\varphi^2)^{\frac{1}{2}}}{(1+\varphi-\varphi^2)^{\frac{1}{2}}} + 1 \\ &= \frac{[(1+\varphi)^{\frac{1}{2}} - (1+\varphi-\varphi^2)^{\frac{1}{2}}][(1+\varphi)^{\frac{1}{2}} + (1+\varphi-\varphi^2)^{\frac{1}{2}}]}{(1+\varphi-\varphi^2)^{\frac{1}{2}}[(1+\varphi)^{\frac{1}{2}} + (1+\varphi-\varphi^2)^{\frac{1}{2}}]} + 1 \\ &= \frac{\varphi^2}{(1+\varphi-\varphi^2)^{\frac{1}{2}}[(1+\varphi)^{\frac{1}{2}} + (1+\varphi-\varphi^2)^{\frac{1}{2}}]} + 1. \end{aligned}$$

Since $(1+\varphi)^{\frac{1}{2}} \geqslant (1+\varphi-\varphi^2)^{\frac{1}{2}}$, we can estimate the denominator above by

$$(1+\varphi-\varphi^2)^{\frac{1}{2}}[(1+\varphi)^{\frac{1}{2}} + (1+\varphi-\varphi^2)^{\frac{1}{2}}] \geqslant 2(1+\varphi-\varphi^2).$$

Thus,

$$E_1(\varphi) \leqslant \frac{\varphi^2}{2(1+\varphi-\varphi^2)} + 1.$$

So

$$\begin{aligned} E(\varphi) &= E_1(\varphi) + E_2(\varphi) \\ &\leqslant \frac{\varphi^2}{2(1+\varphi-\varphi^2)} + 1 - \frac{\varphi^2}{2(1+\varphi)} \\ &= 1 + \frac{\varphi^4}{2(1+\varphi)(1+\varphi-\varphi^2)} \\ &\equiv 1 + \frac{\varphi^4}{D(\varphi)}, \end{aligned} \tag{4.52}$$

where

$$D(\varphi) = 2(1+\varphi)(1+\varphi-\varphi^2).$$

Differentiating, we see that $D'(\varphi) = 4 - 6\varphi^2 > 0$ for $\varphi \in [0, \frac{4}{5}]$. Thus, on the interval $[0, \frac{2}{5}]$ we have $D(\varphi) \geqslant D(0) = 2$, while on the interval $[\frac{2}{5}, \frac{4}{5}]$ we have $D(\varphi) \geqslant D(\frac{2}{5}) = \frac{434}{125}$. Substituting these inequalities into (4.52) we obtain (4.50) and (4.51). $\square$

Lemma 4.4. *For* $\varphi \in [0, \frac{4}{5}]$,

$$F(\varphi) \leqslant \frac{\pi}{4} - \frac{1}{3}\varphi^3. \tag{4.53}$$

Proof. Since $F(0) = \frac{\pi}{4}$ and $F'(\varphi) = \frac{-\varphi^2}{\sqrt{1-\varphi^2}} \leqslant -\varphi^2$, upon integrating we find

$$\begin{aligned} F(\varphi) - F(0) &= \int_0^\varphi F'(s)\, ds \\ &\leqslant \int_0^\varphi -s^2\, ds \\ &= -\frac{1}{3}\varphi^3. \qquad \square \end{aligned}$$

Lemma 4.5. *For* $\varphi \in [0, \frac{4}{5}]$,

$$E(\varphi)F(\varphi) \leqslant \frac{\pi}{4} + \left(\frac{\pi}{16} - \frac{1}{3}\right)\varphi^3. \tag{4.54}$$

Proof. When $\varphi \in [0, \frac{2}{5}]$, by (4.50) and (4.53)

$$\begin{aligned} E(\varphi)F(\varphi) &\leqslant \left(1 + \frac{1}{2}\varphi^4\right)\left(\frac{\pi}{4} - \frac{1}{3}\varphi^3\right) \\ &= \frac{\pi}{4} - \frac{1}{3}\varphi^3 + \frac{\pi}{8}\varphi^4 - \frac{1}{6}\varphi^7 \\ &\leqslant \frac{\pi}{4} - \frac{1}{3}\varphi^3 + \frac{\pi}{8}\varphi^4. \end{aligned}$$

Since $\varphi \leqslant \frac{2}{5}$ implies $\frac{\pi}{8}\varphi^4 \leqslant \frac{\pi}{20}\varphi^3 \leqslant \frac{\pi}{16}\varphi^3$, (4.54) follows.

Similarly, for $\varphi \in \left[\frac{2}{5}, \frac{4}{5}\right]$, by (4.51) and (4.53)

$$\begin{aligned} E(\varphi)F(\varphi) &\leqslant \left(1 + \frac{125}{434}\varphi^4\right)\left(\frac{\pi}{4} - \frac{1}{3}\varphi^3\right) \\ &= \frac{\pi}{4} - \frac{1}{3}\varphi^3 + \frac{125\pi}{(4)(434)}\varphi^4 - \frac{125}{(3)(434)}\varphi^7 \\ &\leqslant \frac{\pi}{4} - \frac{1}{3}\varphi^3 + \frac{125\pi}{(4)(434)}\varphi^4. \end{aligned}$$

When $\varphi \leqslant \frac{4}{5}$, $\frac{125\pi}{(4)(434)}\varphi^4 \leqslant \frac{25\pi}{434}\varphi^3 \leqslant \frac{\pi}{16}\varphi^3$, and once again (4.54) holds. $\square$

Lemma 4.6. *For* $\varphi \in \left[0, \frac{4}{5}\right]$,

$$G(\varphi) \leqslant \frac{\pi}{4} - \frac{1}{2}. \tag{4.55}$$

Proof. Since

$$G'(\varphi) = -\frac{\varphi}{\sqrt{(1-\varphi)(1+\varphi)}} \leqslant 0,$$

it follows that $G(\varphi) \leqslant G(0) = \frac{\pi}{4} - \frac{1}{2}$. $\square$

To complete the proof of Proposition 4.2, notice that (4.49) follows from (4.45), (4.54), and (4.55). $\square$

Combining the results of Propositions 4.1 and 4.2, we see that

$$I_1 + I_2 \leqslant \left(\frac{4}{\pi}\right)\left(\frac{1}{1+\varphi^3}\right)\left\{\frac{\pi}{4} + \left(\frac{\pi}{16} - \frac{1}{3}\right)\varphi^3 + \left(\frac{\pi}{4} - \frac{1}{2}\right)\varphi^3 + \frac{61}{100}\varphi^3\right\}.$$

Therefore,

$$I_1 + I_2 - 1 \leqslant \left(\frac{4}{\pi}\right)\left(\frac{\varphi^3}{1+\varphi^3}\right)\left[\frac{\pi}{16} - \frac{5}{6} + \frac{61}{100}\right] \leqslant 0.$$

It follows that $I_1 + I_2 \leqslant 1$ for $\varphi \in \left[0, \frac{4}{5}\right]$.

4.1.2. Estimate from $\frac{4}{5}$ to 1

When $\varphi \in \left[\frac{4}{5}, 1\right]$, we need to use different estimates of I_1 and I_2. First, recall formula (4.38):

$$\begin{aligned} I_1(\varphi) = \left(\frac{4}{\pi}\right)\left(\frac{1}{1+\varphi^3}\right)\left(\frac{\sqrt{2}}{2}\right)\Bigg[&(2 + 2\varphi^3 - 3\varphi^2)\arcsin\left(\frac{\varphi}{(2+2\varphi^3-3\varphi^2)^{\frac{1}{2}}}\right) \\ &- (2 + 2\varphi^3 - 4\varphi^2)^{\frac{1}{2}}\varphi\Bigg]. \end{aligned}$$

By (4.41)

$$2 - 3\varphi^2 + 2\varphi^3 \geqslant 1,$$

and since arcsin is increasing, it follows that

$$I_1(\varphi) \leqslant \left(\frac{4}{\pi}\right)\left(\frac{1}{1+\varphi^3}\right)\left(\frac{\sqrt{2}}{2}\right)[(2+2\varphi^3-3\varphi^2)\arcsin\varphi - (2+2\varphi^3-4\varphi^2)^{\frac{1}{2}}\varphi]. \quad (4.56)$$

To estimate I_2 we use the fact that $\varphi \leqslant s$ in the integrand in I_2, so

$$s^2 + s - \varphi^3 \geqslant s^2 + s - s\varphi^2 = s(s+1-\varphi^2) \geqslant s^2.$$

Therefore,

$$\begin{aligned} \int_\varphi^1 \frac{s^{5/2}\,ds}{\sqrt{(1-s)(s^2+s-\varphi^3)}} &\leqslant \int_\varphi^1 \frac{s^{3/2}\,ds}{\sqrt{(1-s)}} \leqslant \int_\varphi^1 \frac{s\,ds}{\sqrt{(1-s)}} \\ &= \sqrt{1-s}\left(-\frac{4}{3}-\frac{2}{3}s\right)\Big|_\varphi^1 = \sqrt{1-\varphi}\left(\frac{4}{3}+\frac{2}{3}\varphi\right). \end{aligned}$$

Substituting this into (4.35) we conclude

$$I_2 \leqslant \frac{4}{\pi}\frac{1}{1+\varphi^3}\sqrt{1-\varphi}\left(\frac{4}{3}+\frac{2}{3}\varphi\right). \quad (4.57)$$

Combining (4.56) and (4.57), and observing that $1-2\varphi^2+\varphi^3 = (1-\varphi)(1+\varphi-\varphi^2)$, we obtain

$$\begin{aligned} I_1(\varphi) + I_2(\varphi) \leqslant H(\varphi) \equiv \left(\frac{4}{\pi}\right)\frac{1}{(1+\varphi^3)}\Bigg\{ &\frac{\sqrt{2}}{2}(2-3\varphi^2+2\varphi^3)\arcsin\varphi \\ &+ \sqrt{1-\varphi}\left(-\varphi\sqrt{1+\varphi-\varphi^2}+\frac{4}{3}+\frac{2}{3}\varphi\right)\Bigg\}. \quad (4.58) \end{aligned}$$

It is elementary to estimate that $H\left(\frac{4}{5}\right) < 0.9881 < 1$. We will show that for $\varphi \in \left[\frac{4}{5}, 1\right]$, $H'(\varphi) < 0$, and therefore $I_1(\varphi) + I_2(\varphi) \leqslant H(\varphi) < 1$.

A computation shows that

$$
\begin{aligned}
H'(\varphi) = \frac{1}{\pi}\Bigg\{ & \frac{2\sqrt{2}(-6\varphi+6\varphi^2)\arcsin\varphi}{1+\varphi^3} - \frac{6\sqrt{2}\varphi^2(2-3\varphi^2+2\varphi^3)\arcsin\varphi}{(1+\varphi^3)^2} \\
& - \frac{2\left(\frac{4}{3}+\frac{2\varphi}{3}\right)}{\sqrt{1-\varphi}(1+\varphi^3)} - \frac{2\varphi(-4\varphi+3\varphi^2)}{(1+\varphi^3)\sqrt{1-\varphi}\sqrt{1+\varphi-\varphi^2}} + \frac{2\sqrt{2}(2-3\varphi^2+2\varphi^3)}{\sqrt{1-\varphi^2}(1+\varphi^3)} \\
& \times \frac{-12\sqrt{1-\varphi}\left(\frac{4}{3}+\frac{2\varphi}{3}\right)\varphi^2}{(1+\varphi^3)^2} + \frac{12\varphi^3\sqrt{1-\varphi}\sqrt{1+\varphi-\varphi^2}}{(1+\varphi^3)^2} \\
& + \frac{8\sqrt{1-\varphi}}{3(1+\varphi^3)} - \frac{4\sqrt{1-\varphi}\sqrt{1+\varphi-\varphi^2}}{1+\varphi^3}\Bigg\}.
\end{aligned} \tag{4.59}
$$

To see that $H' < 0$ we will estimate each line of (4.59). First we observe that the arcsin terms simplify to

$$
\frac{6\sqrt{2}\varphi}{\pi(1+\varphi^3)^2}(\varphi^3-2)\arcsin\varphi. \tag{4.60}
$$

Lemma 4.7. *For* $\varphi \in \left[\frac{1}{2}, 1\right]$,

$$
\begin{aligned}
& \frac{1}{\pi}\left(-\frac{2\left(\frac{4}{3}+\frac{2\varphi}{3}\right)}{\sqrt{1-\varphi}(1+\varphi^3)} - \frac{2\varphi(-4\varphi+3\varphi^2)}{(1+\varphi^3)\sqrt{1-\varphi}\sqrt{1+\varphi-\varphi^2}} + \frac{2\sqrt{2}(2-3\varphi^2+2\varphi^3)}{\sqrt{1-\varphi^2}(1+\varphi^3)}\right) \\
& \leqslant \frac{2}{\pi}\frac{1}{(1+\varphi^3)}\left(\frac{2}{3}+2\sqrt{2}\right)\sqrt{1-\varphi}.
\end{aligned}
$$

Proof. We begin by rewriting the left-hand side as

$$
\frac{2}{\pi}\frac{1}{\sqrt{1-\varphi}(1+\varphi^3)}J(\varphi),
$$

where

$$
J(\varphi) = -\frac{4}{3} - \frac{2}{3}\varphi + \frac{\varphi(4\varphi-3\varphi^2)}{\sqrt{1+\varphi-\varphi^2}} + \frac{\sqrt{2}(2-3\varphi^2+2\varphi^3)}{\sqrt{1+\varphi}}.
$$

The polynomial $4\varphi^2 - 3\varphi^3 + \varphi - 2 \leqslant 0$ for $\varphi \in [0, 1]$; therefore,

$$
\frac{\varphi(4\varphi-3\varphi^2)}{\sqrt{1+\varphi-\varphi^2}} \leqslant 4\varphi^2 - 3\varphi^3 \leqslant 2 - \varphi.
$$

Next we use the inequalities

$$\frac{\sqrt{2}}{\sqrt{1+\varphi}} \leqslant 1 + \left(\sqrt{2}-1\right)(1-\varphi), \quad \varphi \in [0,1], \tag{4.61}$$

and

$$2 - 3\varphi^2 + 2\varphi^3 \leqslant 1 + (1-\varphi), \quad \varphi \in \left[\frac{1}{2}, 1\right]. \tag{4.62}$$

To derive these inequalities, simply use the fact that a convex function lies below the line segment between the endpoints. It follows that

$$\begin{aligned}\frac{\sqrt{2}(2-3\varphi^2+2\varphi^3)}{\sqrt{1+\varphi}} &\leqslant \left(1+\left(\sqrt{2}-1\right)(1-\varphi)\right)(1+(1-\varphi)) \\ &\leqslant 1+\sqrt{2}(1-\varphi)+\left(\sqrt{2}-1\right)(1-\varphi)^2 \leqslant 1+\left(2\sqrt{2}-1\right)(1-\varphi).\end{aligned}$$

Combining the preceding estimates, we obtain

$$J(\varphi) \leqslant -\frac{4}{3} - \frac{2}{3}\varphi + 2 - \varphi + 1 + \left(2\sqrt{2}-1\right)(1-\varphi) = \left(\frac{2}{3}+2\sqrt{2}\right)(1-\varphi). \qquad \square$$

Lemma 4.8. *For* $\varphi \in \left[\frac{4}{5}, 1\right]$,

$$\frac{1}{\pi}\left(\frac{-12\sqrt{1-\varphi}\left(\frac{4}{3}+\frac{2\varphi}{3}\right)\varphi^2}{(1+\varphi^3)^2} + \frac{12\varphi^3\sqrt{1-\varphi}\sqrt{1+\varphi-\varphi^2}}{(1+\varphi^3)^2}\right) \leqslant -11\,\frac{\varphi^2\sqrt{1-\varphi}}{\pi(1+\varphi^3)^2}.$$

Proof. Write the left-hand side as

$$\frac{12\varphi^2\sqrt{1-\varphi}}{\pi(1+\varphi^3)^2}K(\varphi),$$

where

$$K(\varphi) = -\frac{4}{3} - \frac{2}{3}\varphi + \varphi\sqrt{1+\varphi-\varphi^2}.$$

It is easy to verify that K is a concave function for $0 \leqslant \varphi \leqslant 1$, and therefore K lies below its tangent line at 1. A computation shows $K'(1) = -\frac{1}{6}$, so $K(\varphi) \leqslant -1 + \frac{1}{6}(1-\varphi)$. Then $K(\varphi) \leqslant -\frac{29}{30} < -\frac{11}{12}$ for $\varphi \geqslant \frac{4}{5}$. $\square$

Lemma 4.9. *For* $\varphi \in [0,1]$,

$$\frac{1}{\pi}\left(\frac{8\sqrt{1-\varphi}}{3(1+\varphi^3)} - \frac{4\sqrt{1-\varphi}\sqrt{1+\varphi-\varphi^2}}{1+\varphi^3}\right) \leqslant -\frac{4}{3\pi}\frac{\sqrt{1-\varphi}}{1+\varphi^3}.$$

Proof. Write the left-hand side as

$$\frac{4}{\pi}\frac{\sqrt{1-\varphi}}{1+\varphi^3}\left(\frac{2}{3} - \sqrt{1+\varphi-\varphi^2}\right).$$

The function $\frac{2}{3} - \sqrt{1+\varphi-\varphi^2}$ is clearly convex for $\varphi \in [0,1]$, therefore it achieves its maximum at the endpoints, where it equals $-\frac{1}{3}$. $\square$

Combining the preceding Lemmas, we have the estimate

$$\begin{aligned} H'(\varphi) &\leqslant \frac{6\sqrt{2}\varphi}{\pi(1+\varphi^3)^2}(\varphi^3-2)\arcsin\varphi + \frac{2}{\pi}\frac{1}{(1+\varphi^3)}\left(\frac{2}{3}+2\sqrt{2}\right)\sqrt{1-\varphi} - 11\frac{\varphi^2\sqrt{1-\varphi}}{\pi(1+\varphi^3)^2} \\ &\quad - \frac{4}{3\pi}\frac{\sqrt{1-\varphi}}{1+\varphi^3} \\ &\leqslant \frac{1}{\pi(1+\varphi^3)^2}\left(6\sqrt{2}\varphi(\varphi^3-2)\arcsin\varphi + \sqrt{1-\varphi}\left(4\sqrt{2}(1+\varphi^3) - 11\varphi^2\right)\right). \end{aligned}$$

The polynomial $4\sqrt{2}(1+\varphi^3) - 11\varphi^2$ is decreasing on $[0,1]$, so

$$4\sqrt{2}(1+\varphi^3) - 11\varphi^2 \leqslant 4\sqrt{2}\left(1+\left(\frac{4}{5}\right)^3\right) - 11\left(\frac{4}{5}\right)^2 < 2 \quad \text{for } \varphi \in \left[\frac{4}{5}, 1\right].$$

Furthermore, $\arcsin\varphi > \frac{5}{6}$ for $\varphi \in \left[\frac{4}{5}, 1\right]$, so we have

$$H'(\varphi) \leqslant \frac{1}{\pi(1+\varphi^3)^2}\left(5\sqrt{2}\varphi(\varphi^3-2)+2\right).$$

A simple calculation shows that the polynomial $\varphi(\varphi^3-2)$ is increasing on $\left[\frac{4}{5}, 1\right]$, so $\varphi(\varphi^3-2) \leqslant -1$ for $\varphi \in \left[\frac{4}{5}, 1\right]$.

Finally, by combining the above estimates we have

$$H'(\varphi) \leqslant \frac{1}{\pi(1+\varphi^3)^2}\left(-5\sqrt{2}+2\right) < 0 \quad \text{for } \varphi \in \left[\frac{4}{5}, 1\right]. \tag{4.63}$$

This completes the proof of Theorem 4.2. $\square$

4.2. $n = 4$

In four dimensions our estimate of the maximal volume is based on the following result of the first author:

Theorem 4.3 (Gursky [14], Theorem B). *If the Yamabe invariant* $Y(M^4, [g]) \geqslant 0$, *then*

$$\int_{M^4} \sigma_2(g^{-1}A)\, dvol = \int_{M^4} \left(-\frac{1}{8}|E|^2 + \frac{1}{96} R^2 \right) dvol \leqslant 4\pi^2. \tag{4.64}$$

Furthermore, equality holds if, and only if, (M^4, g) *is conformally equivalent to the round sphere.*

To prove Theorem 1.4, suppose $g \in \Gamma_k^+(M^4)$ $(k \geqslant 3)$ satisfies

$$\sigma_k^{1/k}(g^{-1}A) \geqslant \sigma_k^{1/k}(S^4).$$

From the Newton–Maclaurin inequality it follows that

$$\sigma_2^{1/2}(g^{-1}A) \geqslant \sigma_2^{1/2}(S^4).$$

Therefore,

$$\begin{aligned} 4\pi^2 &\geqslant \int_{M^4} \sigma_2(g^{-1}A)\, dvol \\ &\geqslant \sigma_2(S^4) vol(M^4, g) \\ &= \frac{3}{2} vol(M^4, g), \end{aligned}$$

and consequently $vol(M^4, g) \leqslant \frac{8}{3}\pi^2 = vol(S^4)$.

Now suppose equality is attained in (1.13). Then there is a sequence of metrics $\{g_j\} \subset \Gamma_k^+(M^4)$ with $\sigma_k^{1/k}(g_j^{-1}A_{g_j}) \geqslant \sigma_k^{1/k}(S^4)$ and $vol(M^4, g_j) \to vol(S^4) = \frac{8}{3}\pi^2$ as $j \to \infty$. Therefore, $\sigma_2^{1/2}(g_j^{-1}A_{g_j}) \geqslant \sigma_2^{1/2}(S^4)$, and appealing once more to (4.64) we have

$$\begin{aligned} 4\pi^2 &\geqslant \int_{M^4} \sigma_2(g_j^{-1}A_{g_j})\, dvol \\ &\geqslant \sigma_2(S^4) vol(M^4, g_j) \to 4\pi^2. \end{aligned}$$

Since $\int \sigma_2$ is conformally invariant, we conclude that

$$\int_{M^4} \sigma_2(g_j^{-1}A_{g_j}) dvol = 4\pi^2$$

for each j. By Theorem 4.3, (M^4, g) is conformally equivalent to the round sphere.

Theorem 1.5 is a consequence of the following estimate:

Theorem 4.4 (Gursky [13], Theorem 1). *If M^4 is a smooth, compact, orientable four-manifold with $b^+ > 0$, then for any metric g of positive scalar curvature the Weyl tensor satisfies*

$$\int_{M^4} |W^+|^2 \, dvol \geqslant \frac{4}{3}\pi^2(2\chi(M^4) + 3\tau(M^4)). \tag{4.65}$$

Furthermore, equality holds if, and only if (M^4, g) is conformal to a positive Kähler–Einstein metric. In this case, M^4 is diffeomorphic to either $S^2 \times S^2$, $\mathbf{CP}^2$, or $\mathbf{CP}^2 \# m(-\mathbf{CP}^2)$ with $3 \leqslant m \leqslant 8$.

Suppose $b^+ > 0$ and $g \in \Gamma_k^+(M^4)$ ($k \geqslant 3$). In particular, this implies that g has positive scalar curvature (i.e., $\sigma_1(g^{-1}A) > 0$). Combining the Chern–Gauss–Bonnet and signature formulas,

$$2\pi^2(2\chi(M^4) + 3\tau(M^4)) = \int_{M^4} |W^+|^2 \, dvol + 2\int_{M^4} \sigma_2(g^{-1}A) \, dvol. \tag{4.66}$$

Combining (4.66) and (4.65), g satisfies

$$\frac{1}{3}\pi^2(2\chi(M^4) + 3\tau(M^4)) \geqslant \int_{M^4} \sigma_2(g^{-1}A) \, dvol.$$

If we normalize g so that

$$\sigma_k^{1/k}(g^{-1}A) \geqslant \sigma_k^{1/k}(S^4),$$

then the Newton–Maclaurin inequality implies

$$\sigma_2^{1/2}(g^{-1}A) \geqslant \sigma_2^{1/2}(S^4).$$

Therefore,

$$\begin{aligned} \frac{1}{3}\pi^2(2\chi(M^4) + 3\tau(M^4)) &\geqslant \int_{M^4} \sigma_2(g^{-1}A) \, dvol \\ &\geqslant \sigma_2(S^4) vol(M^4, g) \\ &= \frac{3}{2} vol(M^4, g), \end{aligned}$$

and it follows that

$$\Lambda_k(M^4, [g]) \leqslant \frac{2}{9}\pi^2(2\chi(M^4) + 3\tau(M^4)).$$

This proves Theorem 1.5.

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